ON THE EXPANSION OF AN ARBITRARY FUNCTION IN TERMS OF LAPLACE'S FUNCTIONS.

By PROF. W. H. ECHOLS, Charlottesville, Va.

1. We recall the following theorem :

If $f(x)$ is a one-valued and continuous function that can be expanded in integral powers of $x - a$ throughout an interval $((\alpha, \beta))$, where $2a = \alpha + \beta$, then can $f(x)$ be expanded in the series

$$f(x) = A_0 + A_1\varphi_1(x) + A_2\varphi_2(x) + \ldots, \tag{1}$$

where $\varphi_r(x)$, $(r = 1, 2, \ldots)$, are rational integral functions of x of degree r constructed according to any assigned law and the coefficients A_r are independent of x. The series (1) being convergent and representing $f(x)$ in $((\alpha, \beta))$.

A demonstration of this consists in designing a rational integral function, with the given $\varphi_r(x)$ functions,

$$A_0 + A_1\varphi_1(x) + \ldots + A_n\varphi_n(x), \tag{2}$$

which shall cut $f(x)$ at $a \equiv a_0$ and at n other arbitrarily chosen points, $a_1, \ldots, a_n$ in $((\alpha, \beta))$, and then making $a_1 = a_2 = \ldots = a_n = a$.

Thus if R be the difference between $f(x)$ and (2) at any point x in $((\alpha, \beta))$ we have

$$f(x) = \sum_0^n A_r\varphi_r(x) + R, \tag{3}$$

and in order that this difference shall be 0 at $a_1, a_2, \ldots, a_n$, we have

$$f(a_j) = \sum_0^n A_r a_j^r, \tag{4}$$

$j = 1, \ldots, n$. Eliminating the A's from (3) and (4), the function (2) is designed. Thus

$$\frac{\begin{vmatrix} f(x), & 1, & \varphi_1(x), & \ldots, & \varphi(x) \\ f(a), & 1, & \varphi_1(a), & \ldots, & \varphi_n(a) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n), & 1, & \varphi_1(a_n), & \ldots, & \varphi_n(a_n) \end{vmatrix}}{\begin{vmatrix} 1, & \varphi_1(a), & \ldots, & \varphi_n(a_n) \end{vmatrix}} = R. \tag{5}$$

But the function R is a one-valued and continuous function of x which admits an indefinite number of one-valued and continuous derivatives in $((\alpha, \beta))$. Since it has the $n+1$ zeros $a_0, \ldots, a_n$, we have for the value of R, by a well known elementary theorem,

$$R = \frac{(x-a_0)\ldots(x-a_n)}{(n+1)!}\left[\frac{d^{n+1}R}{dx^{n+1}}\right]_{x=u},$$

where $\left[\frac{d^{n+1}R}{dx^{n+1}}\right]_{x=u}$, means the $(n+1)$th derivative of R at some point u between the greatest and least of the numbers $x, a_0, \ldots, a_n$. But since the degree of the rational integral functions in the first row is not greater than n, all their $(n+1)$th derivatives are zero, so that

$$R = \frac{(x-a_0)\ldots(x-a_n)}{(n+1)!} f^{n+1}(u),$$

which converges to the limit

$$R = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u),$$

when the a's converge to the limit a, and now u lies between x and a. At the same time the limit of the left side of (5) is evaluated in the usual way by performing on the numerator and denominator the operation

$$\left[\frac{\partial}{\partial a_1}\right]^1_{a_1=a}\left[\frac{\partial}{\partial a_2}\right]^2_{a_2=a}\cdots\left[\frac{\partial}{\partial a_n}\right]^n_{a_n=a}.$$

Whence we have

$$\frac{\begin{vmatrix} f(x), & 1, & \varphi_1(x), & \ldots, & \varphi_n(x) \\ f(a), & 1, & \varphi_1(a), & \ldots, & \varphi_n(a) \\ f'(a), & 0, & \varphi_1'(a), & \ldots, & \varphi_n'(a) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n(a), & 0, & 0, & \ldots, & \varphi_n^n(a) \end{vmatrix}}{\varphi_1'(a)\,\varphi_2''(a)\ldots\varphi_n^n(a)} = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u),$$

or

$$f(x) - \sum_0^n A_r\varphi_r(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{n+1}(u).$$

By hypothesis the limit of the right side is zero, since this is the condition that $f(x)$ can be expanded in powers of $(x - a)$, when $n = \infty$ for all values of x in $((\alpha, \beta))$. By the above construction the coefficients A_r are assigned numbers for any assigned value of the integer r, and we have*

$$f(x) = \sum_0^\infty A_r \varphi_r(x). \tag{6}$$

2. The function

$$\varphi_r(x) = \frac{d^r}{dx^r} \{(x - \alpha)(x - \beta)\}^r,$$

is a rational integral function of degree r, which we write P_x^r for brevity. The function $f(x)$ being as before in § 1, we have

$$f(x) = \sum_0^\infty A_r P_x^r. \tag{7}$$

This function admits of integration between the limits α and β, and we can determine the coefficients A_r in a new form as follows.

The function P_x^r is such that its first r integrals taken from α to β are zero, for the parts independent of the arbitrary constants will contain some power of $(x - \alpha)(x - \beta)$ as a factor.

Write

$$\left\{ \int_\alpha^x dx \int_\alpha^x dx f(x) \right\}_{x=\beta} = F_2,$$

$$\left\{ \int_\alpha^x dx \int_\alpha^x dx P_x^r \right\}_{x=\beta} = P_2^r,$$

with obvious generalization.

Then integrating (7) successively between α and β, we have

$$F_1 = h_1 A_0,$$

$$F_2 = h_2 A_0 + P_2^1 A_1,$$

$$F_3 = h_3 A_0 + P_3^1 A_1 + P_3^2 A_2,$$

$$\cdots\cdots\cdots\cdots\cdots$$

* See Annals of Mathematics, Vol. VII, p. 41.

Wherein for brevity we write $h_r = (\beta - \alpha)^r/r!$. These equations serve to determine the A_r's. Thus

$$A^{r-1} = (-1)^r \frac{\begin{vmatrix} F_1, & h_1, & 0 & 0 & \ldots, & 0 \\ F_2, & h_2, & P_2^1, & 0 & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F_r, & h_r, & P_r^1, & P_r^2, & \ldots, & P_r^{r-2} \end{vmatrix}}{h_1 P_2^1 \ldots P_r^{r-1}}.$$

An assigned number corresponding to an assigned r. This coefficient contains no derivatives of $f(x)$, but is constructed wholly with definite integrals of the functions involved. In particular

$$A_0 = \frac{1}{\beta - \alpha}\int_\alpha^\beta f(x)\,dx$$

is the mean value of $f(x)$ in $((\alpha, \beta))$. We propose to examine this expansion from an entirely independent point of view.

3. The successive integrals of a function $\psi(x)$ can always be expressed by a single integral as follows.

Let

$$\int \psi(x)\,dx = \psi_1(x),$$

then

$$D_x\psi_1(x) = \psi(x).$$

Generally, if

$$\left[\int_\alpha^x dx\right]^r \psi(x) = \psi_r(x),$$

then

$$\left[\int_\alpha^x dx\right]^{r-1} \psi(x) = D_x\psi_r(x) = \psi_{r-1}(x).$$

Now,

$$\int_\alpha^x \psi(x)dx = \psi_1(x) - \psi_1(\alpha),$$

$$\left[\int_\alpha^x dx\right]^2 \psi(x) = \psi_2(x) - \psi_2(\alpha) - (x - \alpha)\,\psi_1(\alpha),$$

and by an easy generalization,

$$\left[\int_a^x dx\right]^n \psi(x) = \psi_n(x) - \sum_0^{n-1} \frac{(x-a)^r}{r!}\psi_{n-r}(a).$$

But

$$\frac{d}{da}\left\{\psi_n(x) - \sum_0^{n-1}\frac{(x-a)^r}{r!}\psi_{n-r}(a)\right\} = -\frac{(x-a)^{n-1}}{(n-1)!}\psi(a).$$

Integrating between a and x, we have

$$\psi_n(x) - \sum_0^{n-1}\frac{(x-a)^r}{r!}\psi_{n-r}(a) = \int_a^x \frac{(x-a)^{n-1}}{(n-1)!}\psi(a)\,da.$$

Since $x - a$ is positive for x in $((\alpha, \beta))$, this can be written

$$\psi(u)\int_a^x \frac{(x-a)^{n-1}}{(n-1)!}\,da = \frac{(x-a)^n}{n!}\psi(u),$$

where u is some number in the interval (a, x).

4. If the first n integrals of a function $\psi(x)$,

$$\psi_r(x) = \left[\int_a^x dx\right]^r \psi(x),$$

$(r = 1, 2, \ldots, n)$ are zero when $x = \beta$, then the function $\psi(x)$ is 0 for n distinct values of x in (α, β). For, $\psi_n(x) = 0$ when $x = \alpha$ and $x = \beta$. Its derivative $\psi_{n-1}(x)$ is therefore zero for some value u_1 between α and β. Therefore, since $\psi_{n-1}(x)$ is zero at α, u_1, β, its derivative $\psi_{n-2}(x)$ is zero for some value of x in each of the intervals (α, u_1), (u_1, β). But $\psi_{n-2}(x)$ is also zero at α and β, or vanishes four distinct times in $((\alpha, \beta))$. Continuing, thus we find that $\psi_1(x)$ is zero at α and β and also at $n - 1$, distinct values between α and β, or altogether $n + 1$ times in the interval $((\alpha, \beta).)$. Therefore the derivative $\psi(x)$ of $\psi_1(x)$ vanishes n distinct times in (α, β).

5. Employing the notations of § 2, the function

$$\frac{\begin{vmatrix} f(x), & 1, & P_x^1, & P_x^2, & \ldots, & P_x^n \\ F_1, & h_1, & 0, & 0, & \ldots, & 0 \\ F_2, & h_2, & P_2^1, & 0, & \ldots, & 0 \\ F_3, & h_3, & P_3^1, & P_3^2, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{n+1}, & h_{n+1}, & P_{n+1}^1, & P_{n+1}^2, & \ldots, & P_{n+1}^n \end{vmatrix}}{h_1 P_2^1 \ldots P_{n+1}^n}, \tag{8}$$

is such that its first $n + 1$ integrals between α and x all vanish for $x = \beta$.* Therefore this function vanishes $n + 1$ times between α and β at, say, $u_1, \ldots, u_{n+1}$. It is therefore equal to

$$\frac{(x - u_1) \ldots (x - u_{n+1})}{(n + 1)!} f^{n+1}(u),$$

where u lies between the greatest and the least of the numbers $x, u_1, \ldots, u_{n+1}$. But

$$\begin{aligned} (x - u_1) \ldots (x - u_{n+1}) \mid &< \mid \beta - \alpha \mid^{n+1}, \\ &= \theta\, (\beta - \alpha)^{n+1}, \end{aligned}$$

where $-1 < \theta < +1$, for all values of x in $((\alpha, \beta))$. Therefore, we have

$$f(x) = \frac{1}{\beta - \alpha} \int_\alpha^\beta f(x) dx + \sum_1^n{}' A_r P_x^r + \theta \frac{(\beta - \alpha)^{n+1}}{(n + 1)!} f^{n+1}(u). \tag{9}$$

If $f(x)$ can be expanded in powers of $(x - \alpha)$ up to and including β, the last term on the right has 0 for its limit when $n = \infty$ and the series

$$f(x) = \frac{1}{\beta - \alpha} \int_\alpha^\beta f(x) dx + \sum_1^\infty{}' A_r P_x^r \tag{10}$$

is convergent and represents $f(x)$ for all values of x in $((\alpha, \beta))$. In the series (10) the coefficients A are independent of x and of derivatives of $f(x)$, being constructed with definite integrals from α to β.

The test for the *convergence* of the series is unfortunately made to depend on the derivatives of $f(x)$. It is desirable that the equivalent of (8) should be evaluated in terms of a definite integral of $f(x)$. The great importance in physics of the expansion of an arbitrary function in terms of Laplace's functions is well known. It is analytically of the greatest importance that the expansion of a one-valued function that is continuous in an interval should be completely solved independent of the operation of differentiation, since there exist such functions whose derivatives are wholly indeterminate and therefore can play no part in the investigation of their properties.†

* As a matter of detail we observe that P_x^n has its n roots real and in (α, β), and also for $r < s$

$$P_{s+m}^{r+m} = P_s.$$

† Annals of Mathematics, Vol. 10, p. 17.